Section 6.6 exercises

1. For each i with $1 \leq i \leq 7$, State i will be the game where Hank has $i-1$ markers and Ted has $7-i$ markers.

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2/3 & 0 & 1/3 & 0 & 0 & 0 & 0 \\ 0 & 2/3 & 0 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 2/3 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 2/3 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

2. For each i with $1 \leq i \leq 9$, State i will be the game where Jessica has $i-1$ markers and John has $9-i$ markers.

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5/16 & 1/8 & 9/16 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5/16 & 1/8 & 9/16 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5/16 & 1/8 & 9/16 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5/16 & 1/8 & 9/16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5/16 & 1/8 & 9/16 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5/16 & 1/8 & 9/16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5/16 & 1/8 & 9/16 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

3. Define states as follows:

State 1	Andre wins	State 4	Andre up 1	State 7	Pete up 2
State 2	Andre up 3	State 5	Tied	State 8	Pete up 3
State 3	Andre up 2	State 6	Pete up 1	State 9	Pete wins

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2/3 & 0 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2/3 & 0 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2/3 & 0 & 1/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2/3 & 0 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2/3 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2/3 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

6. States 1 and 2 refer to a game piece being on squares A and B, respectively. The transition matrix is

$$M = \begin{bmatrix} 5/6 & 1/6 \\ 0 & 1 \end{bmatrix}$$

This game is played by rolling a die until a result of "6" occurs for the first time. This is just like Exercise 1 in Section 6.5.

12. The probability of going from State 5 to State 7 in no more than 16 moves is approximately 0.232, since this is the entry in Row 5, Column 7 of the matrix

$$M^{16} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.971 & 0.0065 & 0 & 0.0065 & 0 & 0.0016 & 0.014 \\ 0.926 & 0 & 0.020 & 0 & 0.0098 & 0 & 0.044 \\ 0.837 & 0.026 & 0 & 0.026 & 0 & 0.0065 & 0.105 \\ 0.710 & 0 & 0.039 & 0 & 0.020 & 0 & 0.232 \\ 0.456 & 0.026 & 0 & 0.026 & 0 & 0.0065 & 0.486 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

13. The probability of going from State 5 to State 1 in no more than 16 moves is approximately 0.0642, since this is the entry in Row 5, Column 1 of the matrix

$$M^{16} = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.530 & 0.00741 & 0.0178 & 0.0306 & 0.0423 & 0.0510 & 0.0501 & 0.0359 & 0.235 \\ 0.273 & 0.00990 & 0.0244 & 0.0413 & 0.0590 & 0.0702 & 0.0710 & 0.0501 & 0.401 \\ 0.135 & 0.00945 & 0.0230 & 0.0402 & 0.0568 & 0.0701 & 0.0702 & 0.0510 & 0.544 \\ 0.0642 & 0.00726 & 0.0182 & 0.0316 & 0.0463 & 0.0568 & 0.0590 & 0.0423 & 0.674 \\ 0.0288 & 0.00486 & 0.0120 & 0.0216 & 0.0316 & 0.0402 & 0.0413 & 0.0306 & 0.789 \\ 0.0118 & 0.00265 & 0.00676 & 0.0120 & 0.0182 & 0.0230 & 0.0244 & 0.0178 & 0.883 \\ 0.00384 & 0.00106 & 0.00265 & 0.00486 & 0.00726 & 0.00945 & 0.00990 & 0.00741 & 0.954 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}$$

14. The probability of going from State 5 to State 1 in no more than 16 moves is approximately 0.798, since this is the entry in Row 5, Column 1 of the matrix

$$M^{16} = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.973 & 0.00842 & 0.0 & 0.00990 & 0.0 & 0.00476 & 0.0 & 0.000957 & 0.00256 \\ 0.937 & 0.0 & 0.0282 & 0.0 & 0.0194 & 0.0 & 0.00667 & 0.0 & 0.00864 \\ 0.864 & 0.0396 & 0.0 & 0.0472 & 0.0 & 0.0232 & 0.0 & 0.00476 & 0.0208 \\ 0.798 & 0.0 & 0.0776 & 0.0 & 0.0549 & 0.0 & 0.0194 & 0.0 & 0.0499 \\ 0.666 & 0.0761 & 0.0 & 0.0930 & 0.0 & 0.0472 & 0.0 & 0.00990 & 0.108 \\ 0.553 & 0.0 & 0.107 & 0.0 & 0.0776 & 0.0 & 0.0282 & 0.0 & 0.234 \\ 0.328 & 0.0613 & 0.0 & 0.0761 & 0.0 & 0.0396 & 0.0 & 0.00842 & 0.487 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}$$

24. We use the matrix

$$N = \begin{bmatrix} 0 & 1/3 & 0 & 0 & 0 \\ 2/3 & 0 & 1/3 & 0 & 0 \\ 0 & 2/3 & 0 & 1/3 & 0 \\ 0 & 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 0 & 2/3 & 0 \end{bmatrix}$$

corresponding to the transient states from the matrix given in Exercise 1, and compute

$$N + N^2 + \cdots + N^{1000} \approx \begin{bmatrix} 0.476 & 0.714 & 0.333 & 0.143 & 0.0476 \\ 1.43 & 1.14 & 1.00 & 0.429 & 0.143 \\ 1.33 & 2.00 & 1.33 & 1.00 & 0.333 \\ 1.14 & 1.71 & 2.00 & 1.14 & 0.714 \\ 0.762 & 1.14 & 1.33 & 1.43 & 0.476 \end{bmatrix}$$

Adding 1 to the sum of the entries in the fourth row (corresponding to games beginning in State 5) gives us approximately 7.71 coin tosses expected in this game.

25. We use the matrix

$$N = \begin{bmatrix} 1/8 & 9/16 & 0 & 0 & 0 & 0 & 0 \\ 5/16 & 1/8 & 9/16 & 0 & 0 & 0 & 0 \\ 0 & 5/16 & 1/8 & 9/16 & 0 & 0 & 0 \\ 0 & 0 & 5/16 & 1/8 & 9/16 & 0 & 0 \\ 0 & 0 & 0 & 5/16 & 1/8 & 9/16 & 0 \\ 0 & 0 & 0 & 0 & 5/16 & 1/8 & 9/16 \\ 0 & 0 & 0 & 0 & 0 & 5/16 & 1/8 \end{bmatrix}$$

corresponding to the transient states from the matrix given in Exercise 2, and compute

$$N + N^2 + \cdots + N^{1000} \approx \begin{bmatrix} 0.765 & 1.74 & 1.70 & 1.62 & 1.49 & 1.24 & 0.797 \\ 0.967 & 1.71 & 2.64 & 2.52 & 2.31 & 1.93 & 1.24 \\ 0.524 & 1.47 & 2.17 & 3.03 & 2.77 & 2.31 & 1.49 \\ 0.278 & 0.779 & 1.68 & 2.30 & 3.03 & 2.52 & 1.62 \\ 0.142 & 0.396 & 0.855 & 1.68 & 2.17 & 2.64 & 1.70 \\ 0.0656 & 0.184 & 0.396 & 0.779 & 1.47 & 1.71 & 1.74 \\ 0.0234 & 0.0656 & 0.142 & 0.278 & 0.524 & 0.967 & 0.765 \end{bmatrix}$$

Adding 1 to the sum of the entries in the fourth row (corresponding to games beginning in State 5) gives us approximately 13.21 moves expected in this game.

26. We use the matrix

$$N = \begin{bmatrix} 0 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 2/3 & 0 & 1/3 & 0 & 0 & 0 & 0 \\ 0 & 2/3 & 0 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 2/3 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 2/3 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 2/3 & 0 \end{bmatrix}$$

corresponding to the transient states from the matrix given in Exercise 3, and compute

$$N + N^2 + \cdots + N^{1000} \approx \begin{bmatrix} 0.494 & 0.741 & 0.365 & 0.176 & 0.0824 & 0.0353 & 0.0118 \\ 1.48 & 1.22 & 1.09 & 0.529 & 0.247 & 0.106 & 0.0353 \\ 1.46 & 2.19 & 1.55 & 1.24 & 0.576 & 0.247 & 0.0824 \\ 1.41 & 2.12 & 2.47 & 1.65 & 1.24 & 0.529 & 0.176 \\ 1.32 & 1.98 & 2.31 & 2.47 & 1.55 & 1.09 & 0.365 \\ 1.13 & 1.69 & 1.98 & 2.12 & 2.19 & 1.22 & 0.741 \\ 0.753 & 1.13 & 1.32 & 1.41 & 1.46 & 1.48 & 0.494 \end{bmatrix}$$

Adding 1 to the sum of the entries in the fourth row (corresponding to games beginning in State 5) gives us approximately 10.6 moves expected in this game.

29. We use the matrix

$$N = \begin{bmatrix} 5/6 \end{bmatrix}$$

corresponding to the transient states from the matrix given in Exercise 6, and compute

$$N + N^2 + \cdots + N^{1000} \approx \begin{bmatrix} 5.0 \end{bmatrix}$$

Adding 1 to the sum of the entries in the first row (corresponding to games beginning in State 1) gives us approximately 6.0 moves expected in this game.